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Band radiation of isothermal gases within diffuse-walled enclosures

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Abstract—A method for obtaining highly accurate approximate solutions of the spectrally integrated, zonal, radiative transfer equations for arbitrary enclosures containing a nongray, isothermal gas is developed in this paper. The procedure is general enough to be applied to any nongray absorption–emission spectrum but is particularly appropriate for band radiation of molecular gases. It represents a fundamentally based alternative to presently available gray-band or gray-gas procedures.

INTRODUCTION

THE ANALYSIS of radiative transfer within diffuse-walled enclosures has a long and interesting history. The early work of Hottel [1] laid the foundations of a workable theory based upon a limited knowledge of gas radiative properties. Approximations were devised and applied with remarkable success to problems which had previously been intractable. The emphasis of this early work was upon design procedures for industrial furnaces.

More fundamentally based studies of radiative transfer processes in the engineering context began to evolve with the work of Bevans and Dunkle [2] and somewhat later by that of Edwards [3]. These later workers recognized that many problems of practical interest could not be adequately solved based solely upon knowledge of isothermal gas emissivities and absorptivities. They relied instead upon the more fundamental band—absorption properties of gases, which preserved some essential spectral detail and whose empirical representation had begun to appear with the experimental work of Howard *et al.* [4] and Edwards [5]. The work of Bevans and Dunkle [2] and its later clarification by Edwards [3], however, relied heavily upon the network method developed by Oppenheim [6] and the gray-band concept of an effective band width.

In a recent survey of nongray radiative transfer from gases, Edwards [7] proposed some generalized rules for evaluating the effective band width or its equivalent but these are largely the product of intuition and are not directly derived from fundamental principles.

For some problems of interest the approach of Bevans–Dunkle–Edwards (BDE) is superior to that of Hottel but the latter is of greater generality since it is not restricted to the isothermal gas assumption. The BDE method, however, treats nongray gas radiation properties more rigorously than that of Hottel, the later of which is a modified gray-band model and consequently has many of its well-known shortcomings.

The purpose of this present paper is to develop a band-radiation method of general applicability for enclosures of arbitrary geometry with a minimum of special assumptions.

Hottel's zone method [1, 8] is familiar to the specialist and nonspecialist alike and also is a convenient framework for the more general nongray methodology which will be developed in this paper.

In an earlier work [9] a closure method was devised and applied to very simple problems. The same basic approach will be applied herein to the general enclosure of fundamental engineering interest.

In this development it is not a variation of the zone method which is discussed but rather the way in which nongray radiation is treated within the context of the zone method.

ANALYSIS

When an enclosure is idealized through the assumption of an isothermal gas, the basic transfer relations for the volumetric cooling rate of the gas and the total radiosity for a surface can be somewhat simplified.

The general, band-radiation zone equations were given in ref. [9] and when one makes use of equivalence relations between exchange factors they are conveniently written for an isothermal gas as follows (Appendix):

Gas cooling rate

$$Q^{*\mathrm{r}} = \sum_{k=1}^{K} \sum_{t=1}^{N} \sum_{i=1}^{N} \tilde{A}_i \omega_k S a_{kit} [E_{kb}(T_{\mathrm{g}}) - R_{ki}^{(t)}]; \quad (1)$$

total surface radiosity

$$R_t = \varepsilon_t E_{\mathrm{b}}(T_t) + \rho_t \sum_{k=1}^{K} \sum_{i=1}^{N} \omega_k S a_{kti} [E_{k\mathrm{b}}(T_{\mathrm{g}}) - R_{ki}^{(t)}] + \rho_t \sum_{i=1}^{N} F_{ti} R_i \quad (2)$$

for a known surface temperature, T_t or

$$R_t = q_t^{\mathrm{r}} + \sum_{k=1}^{K} \sum_{i=1}^{N} \omega_k S a_{kti} [E_{k\mathrm{b}}(T_{\mathrm{g}}) - R_{ki}^{(t)}] + \sum_{i=1}^{N} F_{ti} R_i \quad (3)$$

if q_t^{r} is given, either directly or through a surface energy balance.

NOMENCLATURE

$A(x)$	band absorption function of argument x
$\tilde{A}$	surface area
E	radiative emissive power
F	surface-to-surface shape factor
H	radiative surface irradiation
N	index for the maximum number of surfaces
q^{r}	radiative heat flux at a surface
$Q^{*\mathrm{r}}$	gas radiative cooling rate
r	linear distance, generally between two points in space
R	surface radiosity
S	gas transmissivity between surfaces
Sa	gas absorption function between surfaces
T	temperature
u	dimensionless mean beam length.

Greek symbols

δ_{ij}	Kronecker delta function
ε	surface emissivity
θ	angle measured from a surface normal
ν	radiation wavenumber
ρ	surface reflectivity
σ	Stefan–Boltzmann constant
τ	optical depth
ψ	equation (28) and following text
ω	band-width parameter.

Subscripts

b	black body
g	gas phase
k	specific gas absorption band
ν	spectral quantity
all others	generally refer to a surface zone.

Superscripts

all	refer to surface zones or if (g) to the gas zone.

The band radiosity function $R_{ki}^{(t)}$ is defined by [9]

$$Sa_{kti}R_{ki}^{(t)} = \int_{\Delta\nu_k} Sa_{\nu ti}R_{\nu i}\frac{\mathrm{d}\nu}{\omega_k}. \tag{4}$$

Band radiosity equation

The spectral radiosity equation for a general enclosure containing an isothermal gas is

$$R_{\nu i}-E_{\nu\mathrm{b}}(T_{\mathrm{g}}) = \varepsilon_i[E_{\nu\mathrm{b}}(T_i)-E_{\nu\mathrm{b}}(T_{\mathrm{g}})] + \rho_i\sum_{j=1}^{N} S_{\nu ij}[R_{\nu j}-E_{\nu\mathrm{b}}(T_{\mathrm{g}})]. \tag{5}$$

Multiplication of equation (5) by $Sa_{\nu ti}\,\mathrm{d}\nu/\omega_k$ and integration over $\Delta\nu_k$ gives

$$R_{ki}^{(t)}-E_{k\mathrm{b}}(T_{\mathrm{g}}) = \varepsilon_i[E_{k\mathrm{b}}(T_i)-\mathrm{E}_{k\mathrm{b}}(T_{\mathrm{g}})] + \rho_i\sum_{j=1}^{N} S_{kij}^{(t)}[R_{kj}^{(it)}-E_{k\mathrm{b}}(T_{\mathrm{g}})] \tag{6}$$

where

$$Sa_{kti}S_{kij}^{(t)}R_{kj}^{(it)} = \int_{\Delta\nu_k} Sa_{\nu ti}S_{\nu ij}R_{\nu j}\frac{\mathrm{d}\nu}{\omega_k} \tag{7}$$

and

$$S_{kij}^{(t)} = \frac{1}{Sa_{kti}}\int_{\Delta\nu_k} Sa_{\nu ti}S_{\nu ij}\frac{\mathrm{d}\nu}{\omega_k}. \tag{8}$$

From equations (5) and (7) one can find by the same procedure that

$$R_{kj}^{(it)}-E_{k\mathrm{b}}(T_{\mathrm{g}}) = \varepsilon_j[E_{k\mathrm{b}}(T_j)-E_{k\mathrm{b}}(T_{\mathrm{g}})] + \rho_j\sum_{m=1}^{N} S_{kjm}^{(it)}[R_{km}^{(jit)}-E_{k\mathrm{b}}(T_{\mathrm{g}})] \tag{9}$$

with

$$S_{kjm}^{(it)} = \frac{1}{Sa_{kti}S_{kij}^{(t)}}\int_{\Delta\nu_k} Sa_{\nu ti}S_{\nu ij}S_{\nu jm}\frac{\mathrm{d}\nu}{\omega_k} \tag{10}$$

and an obvious definition of $R_{km}^{(jit)}$.

The procedure illustrated above could be continued indefinitely and in the process one would generate an infinite series solution for $R_{ki}^{(t)}$ by back substitution of each higher-order equation into the next lower order. The structure of that solution is of interest and is outlined in the following.

Structure of the exact solution. One will obtain the exact solutions for the band radiosities by back substitution of higher-order into lower-order equations. An expansion to three explicit terms yields

$$\begin{aligned} R_{ki}^{(t)}-E_{k\mathrm{b}}(T_{\mathrm{g}}) &= \varepsilon_i[E_{k\mathrm{b}}(T_i)-E_{k\mathrm{b}}(T_{\mathrm{g}})] \\ &\quad+\rho_i\sum_{j=1}^{N} S_{kij}^{(t)}\varepsilon_j[E_{k\mathrm{b}}(T_j)-E_{k\mathrm{b}}(T_{\mathrm{g}})] \\ &\quad+\rho_i\sum_{j=1}^{N}\sum_{m=1}^{N}\rho_j S_{kij}^{(t)}S_{kjm}^{(it)}\varepsilon_m[E_{k\mathrm{b}}(T_m)-E_{k\mathrm{b}}(T_{\mathrm{g}})] \\ &\quad+\rho_i\sum_{j=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N}\rho_j\rho_m S_{kij}^{(t)}S_{kjm}^{(it)}S_{kmn}^{(jit)} \\ &\quad\times[R_{kn}^{(mjit)}-E_{k\mathrm{b}}(T_{\mathrm{g}})]. \end{aligned} \tag{11}$$

The physical interpretation of these terms is easy. The first is simply the emission component of the radiosity, the second represents radiation which has been received directly and is then reflected, the third accounts for radiation received after having undergone one reflection prior to reflection from surface, i, the last term takes into account radiation which has been reflected two or more times prior to incidence on surface, i. The last term is clearly not explicit and can only be made so by continuing the term-by-term expansion. The end result is an infinite series solution which describes all possible combinations of multiple reflections term by term.

For nongray radiation this form of expansion appears essential to obtain an exact solution, because, as indicated by superscripting, the spectrally integrated transmission of a band is history dependent.

Only for spectral or gray-band radiation is it possible to ignore prior history, which amounts in this formalism to dropping superscripts.

Although the infinite series solution has a number of valuable uses it hardly fills the need for a practical method of enclosure analysis which could be applied on a routine basis. To achieve such an objective an approximate procedure is described in the following.

A closure approximation. Approximate solutions of the band radiosity equations were obtained in ref. [9] for a particularly simple example problem. Closure was obtained by simply deleting a superscript from the band-radiosity functions of higher order appearing in any given band-radiosity equation. Because only one surface zone was considered there was no difficulty in selecting a superscript to be deleted.

When multiple surface zones are involved a band-radiosity equation of arbitrary order can be written for an isothermal gas as

$$R_{ki}^{(lmn\ldots st)} - E_{kb}(T_g) = \varepsilon_i[E_{kb}(T_i) - E_{kb}(T_g)] + \rho_i \sum_{j=1}^{N} S_{kij}^{(lmn\ldots st)}[R_{kj}^{(ilmn\ldots st)} - E_{kb}(T_g)]. \quad (12)$$

This equation is not closed because the band-radiosity function on the right is of higher order than that on the left.

It is not possible to arbitrarily select a closure method without the possibility of encountering difficulties which are not easily resolved in a satisfactory way. It has been found that the only closure method which is free of fundamental problems is to put

$$R_{kj}^{(ilmn\ldots st)} \approx R_{kj}^{(ilmn\ldots s)} \quad (13)$$

so that equation (12) becomes, in the closure approximation

$$R_{ki}^{(lmn\ldots st)} - E_{kb}(T_g) = \varepsilon_i[E_{kb}(T_i) - E_{kb}(T_g)] + \rho_i \sum_{j=1}^{N} S_{kij}^{(lmn\ldots st)}[R_{kj}^{(ilmn\ldots s)} - E_{kb}(T_g)]. \quad (14)$$

This equation is closed because the band-radiosity functions are all of the same order. A separate equation holds for each possible combination of the superscripts but some of these are physically irrelevant, as discussed below.

When dealing with band radiosities one must be concerned with physically meaningful radiosity functions. The sequence of subscript/superscripts physically represents the path followed by the radiation contributing to the radiosity function of interest. Some paths will be physically unacceptable and these need to be recognized at the outset so they can be excluded from the system of band-radiosity equations.

Unacceptable paths always involve the transfer of radiation from a surface to another part of itself or to another surface which is shielded by an intervening object. The second possibility is excluded from consideration at this time. The first is possible only if the considered surface is concave, thus in general the unacceptable paths of interest will always involve a convex surface which cannot receive radiation from itself. A typical unacceptable path would be represented by a radiosity function of the form $R_{ki}^{(lmm\ldots st)}$ where surface, m, is convex. The repeated superscript could occur anywhere within the superscript sequence. Another unacceptable form would be $R_{ki}^{(imn\ldots st)}$ with, i, convex. These physically meaningless radiosity functions must be excluded from the basic system of relevant radiosity equations. The exclusion can be direct or indirect. The mechanism for the indirect approach might be simpler and will be discussed further on.

That equation (13) is the only fundamentally acceptable closure assumption can be explained as follows. Suppose that instead of equation (13) one puts $R_{kj}^{(ilmn\ldots st)} \approx R_{kj}^{(lmn\ldots st)}$ then the resultant form of equation (12) is actually simpler than equation (14) because of a lack of superscript coupling, namely, equations with different superscript sequences are independent of one another. Unfortunately, the inverse matrix coefficients will always contain one row with elements containing transmissivities of the form $S_{kij}^{(imn\ldots st)}$ in which an interior member of the sequence $(jiimn\ldots st)$ is repeated. When this happens the band transmissivity for a convex surface, i, is formally indeterminant, being 0/0. There apparently is no fundamental way of removing this difficulty and although ad-hoc plausibility arguments can be devised to deal with the problem, an otherwise fundamentally based methodology becomes linked to an essentially arbitrary rule for assigning values to indeterminant quantities. Furthermore, one cannot choose a value of zero because that is inconsistent with a gray-band model and leads to a complete neglect of a number of multiple reflections. In the case of a planar medium all multiply reflected radiation is neglected. Closure rule (13) is the only one free of these problems.

Equation (13) is mathematically motivated and without a clear physical interpretation of its own, however, the result of this approximation does have a clear physical interpretation and is discussed later.

The closure method is very powerful and the best way to illustrate this is through problems that can be analytically solved. Consequently, only low-order solutions for two-surface-zone enclosures will be quantitatively considered. More complex, multisurfaced enclosures will, however, be fully discussed.

Structure of the approximate solution. An infinite series solution for equation (14) for any given order of closure can be constructed in exactly the same way as was done for the exact formulation. The physical meaning of the closure approximation then becomes quite clear.

In the exact expansion one finds that radiation which has been reflected n times is attenuated by a product of band transmissivities which vary in order from 1 to n. In a first-order closure one finds instead a product of n first-order band transmissivities; at second order the transmissivity product consists of a first order followed by $(n-1)$ of second order; at third order one gets a first order, a second order and $(n-2)$ of third and so on up to any order desired less than or equal to n. Consequently, a closure approximation of order n treats up to n reflections exactly and approximates all higher-order reflections by attenuating them with band transmissivities of order n.

The great advantage of the closure method is that one does not have to carry out a series expansion, as must be done to obtain an exact result. Rather, the closed-band-radiosity equations of a given order can be solved by matrix inversion or other procedures, such as Gaussian elimination, with all the usual computational benefits.

Moreover, it will be shown herein, as it was elsewhere [9], that relatively low order approximations are nearly always completely satisfactory, but before discussing specific examples the evaluation of the surface total heat flux or temperature is discussed.

Evaluation of radiative flux or surface temperature. When the band radiosities are determined the total radiosity can be found from equation (2). This makes it possible to find either the total surface heat flux, when the temperature is prescribed, or the surface temperature, when the total heat flux is given. When the heat flux is sought it is found from

$$q_t^{\mathrm{r}} = \frac{\varepsilon_t}{\rho_t}\,[E_{\mathrm{b}}(T_t) - R_t]; \quad \rho_t \neq 0 \tag{15}$$

or

$$q_t^{\mathrm{r}} = E_{\mathrm{b}}(T_t) - H_t; \quad \rho_t = 0 \tag{16}$$

where

$$H_t = \sum_{k=1}^{K} \sum_{i=1}^{N} \omega_k\, Sa_{kti}[E_{k\mathrm{b}}(T_{\mathrm{g}}) - R_{ki}^{(t)}] + \sum_{i=1}^{N} F_{ti}R_i \tag{17}$$

is the total surface irradiation.

When the heat flux is given, equation (15) is simply rearranged, using $E_{\mathrm{b}}(T) = \sigma T^4$, to give

$$T_t = \left[\frac{1}{\sigma}(R_t + \rho_t q_t^{\mathrm{r}}/\varepsilon_t)\right]^{1/4}; \quad \rho_t \neq 1. \tag{18}$$

If by chance, $\rho_t = 1$, the surface temperature cannot be found from the radiative transfer analysis alone.

Equation (18) for the surface temperature can only give an iterated value since when solving the band-radiosity equations only prescribed surface temperatures can be used. This is because no physical principle can prescribe band radiative fluxes from a knowledge of total flux alone. Estimated surface temperatures, however, are only used when solving for band radiosities whereas any given total flux should always be used in the total radiosity equations. This is expected to speed convergence of the overall solution.

The planar medium

The planar medium eliminates all unnecessary complications while at the same time allowing two distinct surface zones to be defined. Cylindrical annuli or spherical shells also have this later property but introduce geometric complexity.

Solving the system of equations for the radiative flux, the total radiosity and the band radiosities yields for the flux at surface 1 the first-order relation

$$\begin{aligned} q_1^{\mathrm{r}} = \frac{\varepsilon_1\varepsilon_2}{1-\rho_1\rho_2} \Bigg\{ &[E_{\mathrm{b}}(T_1) - E_{\mathrm{b}}(T_2)] - \sum_{k=1}^{K} \omega_k\, Sa_{k12} \\ &\times [E_{k\mathrm{b}}(T_1) - E_{k\mathrm{b}}(T_2)]\, \frac{1+\rho_1\rho_2 S_{k21}^{(1)}}{1-\rho_1\rho_2 S_{k21}^{(1)} S_{k12}^{(2)}} \Bigg\} \\ &+ \varepsilon_1 \sum_{k=1}^{K} \omega_k\, Sa_{k12}[E_{k\mathrm{b}}(T_1) - E_{k\mathrm{b}}(T_{\mathrm{g}})] \\ &\times \frac{1+\rho_2 S_{k21}^{(1)}}{1-\rho_1\rho_2 S_{k21}^{(1)} S_{k12}^{(2)}} \end{aligned} \tag{19}$$

and where the relation for q_2^{r} is simply obtained by exchanging all (1, 2) indices to (2, 1). It might be noted that only the sign of the first two terms is affected but that the last term might change significantly in magnitude. The physical interpretation of the above equation is difficult simply because the mathematical manipulations frequently arrange terms in a way which is not easy to interpret. The form of this equation has required algebraic manipulations of the basic results using obvious symmetry relations.

Equation (19) can be reduced to that found in an earlier paper [9] by putting $\varepsilon_1 = \varepsilon_2$, $T_2 = T_1$, using symmetry to put $S_{k21}^{(1)} = S_{k12}^{(2)}$ and noting that $1-[\rho S_{k12}^{(2)}]^2 = [1+\rho S_{k12}^{(2)}][1-\rho S_{k12}^{(2)}]$. This reduction is significant because the present results have been obtained by solving the surface radiosity equations while the former were obtained by solving for the gas cooling rate.

There are other interesting variants of equation (19) related specifically to problems studied in earlier works. If one puts $\varepsilon_1 = 1$ then the problem considered by Edwards [3] results. It is of interest because Edwards was the first to consider, in detail, the exact solution for a planar reflecting enclosure containing a band absorbing medium. Furthermore, if one also puts $\rho_2 = 1$ then a special case of one adiabatic wall also results.

The most generally interesting case occurs when $q_2 = 0$ since this is the simplest model of many industrial furnace designs, as pointed out by Hottel and Sarofim [10] and Hottel [11]. The temperature of surface 2 must then be found by iteration from

$$E_b(T_2)-E_b(T_1) = \sum_{k=1}^{K} \omega_k Sa_{k12}[E_{kb}(T_2)-E_{kb}(T_1)] \times \frac{1+\rho_1\rho_2 S^{(1)}_{k21}}{1-\rho_1\rho_2 S^{(1)}_{k21} S^{(1)}_{k21}} - \frac{1-\rho_1\rho_2}{\varepsilon_1} \sum_{k=1}^{K} \times \omega_k Sa_{k12}[E_{kb}(T_2)-E_{kb}(T_g)] \frac{1+\rho_1 S^{(1)}_{k21}}{1-\rho_1\rho_2 S^{(1)}_{k21} S^{(1)}_{k21}} \quad (20)$$

with the flux at surface 1 given by equation (19).

An interesting case occurs if $\varepsilon_1 = 1$ for then

$$E_b(T_2) = E_b(T_1) + \sum_{k=1}^{K} \omega_k Sa_{k12}[E_{kb}(T_g)-E_{kb}(T_1)] \quad (21)$$

which requires no iterations. The flux for this case is given by

$$q^r_1 = \varepsilon_2 \sum_{k=1}^{K} \omega_k Sa_{k12}[E_{kb}(T_2)-E_{kb}(T_g)] + \sum_{k=1}^{K} \omega_k Sa_{k12}[E_{kb}(T_1)-E_{kb}(T_g)][1+\rho_2 S^{(1)}_{k21}]. \quad (22)$$

When surface 2 as well as surface 1 is black one has the completely black wall enclosure with one adiabatic wall for which the flux is given by

$$q^r_1 = \sum_{k=1}^{K} \omega_k Sa_{k12}[E_{kb}(T_1)+E_{kb}(T_2)-2E_{kb}(T_g)]. \quad (23)$$

If, however, surface 2 is perfectly reflecting while surface 1 is black then

$$q^r_1 = \sum_{k=1}^{K} \omega_k Sa_{k12}[E_{kb}(T_1)-E_{kb}(T_g)][1+S^{(1)}_{k21}] \quad (24)$$

which is frequently considerably different from the previous result, especially at large optical depth, as shown in Table 1 for a fixed gas temperature. The physical reasons for the difference should be clear.

Equation (19) has been solved to first through fourth order for a wide combination of ρ_1 and ρ_2 and for the special case of equal wall temperatures. These solutions correspond to a three-zone counterpart of the two-zone example worked out in ref. [9]. The results, which are too numerous to give here, confirm the conclusion drawn from the earlier work, namely, first-order solutions are nearly always completely acceptable. As before, higher-order solutions might be desired only when the average reflectivity of the enclosure is 0.7 or larger. Such high values will rarely be encountered in practice for they defeat the usual objective of maximizing the radiative flux. Additional confirmation is given for a more complex enclosure further on.

Table 1. Heat transfer from a planar medium with one adiabatic wall

Optical depth	Heat transfer ratio† $\rho_2 = 0$	$\rho_2 = 1$
0.1	0.9985	0.9572
1.0	0.9905	0.7757
10.0	0.9738	0.6110
100.0	0.9575	0.5660

† The results are the ratio of net heat transfer from the gas when one wall is adiabatic and the other black to that when both walls are black and at equal temperatures: $T_g = 2000$ K, $T_w = 800$ K, $\rho_1 = 0$, band at 2410 cm^{-1}, $\omega_k = 50$ cm^{-1}.

The three-zone enclosure

The most general three-zone enclosure consists of the gas zone and two concave surface zones. A first-order closure, therefore, requires a solution for four physically acceptable band radiosities, $R^{(1)}_{k1}$, $R^{(2)}_{k1}$, $R^{(1)}_{k2}$ and $R^{(2)}_{k2}$. The existence of six zero elements makes it rather easy to determine the inverse matrix, which contains 16 non-zero elements. However, in expressing the solution for the band radiosities these inverse matrix elements combine in groups of two leaving a total of eight coefficients which determine the solution in the general form

$$R^{(t)}_{ki} - E_{kb}(T_g) = \sum_{j=1}^{2} \varepsilon_j \psi^{(t)}_{kij}[E_{kb}(T_j)-E_{kb}(T_g)]. \quad (25)$$

The coefficients $\psi^{(t)}_{kij}$ satisfy, in this case, a symmetry condition which makes it necessary to give explicit equations for only four of them, these are given in the appendix.

The second-order solution can be similarly expressed as

$$R^{(st)}_{ki} - E_{kb}(T_g) = \sum_{j=1}^{2} \varepsilon_j \psi^{(st)}_{kij}[E_{kb}(T_j)-E_{kb}(T_g)] \quad (26)$$

but there are now 16 $\psi^{(st)}_{kij}$ obtained from an inverse matrix containing 64 elements; each of the $\psi^{(st)}_{kij}$ consists of four elements of the inverse matrix. This solution is, clearly, best carried out on a computer or advanced programmable calculator.

The solution for the radiative flux at either surface is of importance and can be expressed as

$$q^r_j = \frac{\varepsilon_i\varepsilon_j}{\rho_j} \Lambda_{ji}[E_b(T_j)-E_b(T_i)] + \frac{\varepsilon_j}{\rho_j} \sum_{k=1}^{K} \sum_{m=1}^{2} \omega_k \phi_{kjm} \varepsilon_m [E_{kb}(T_m)-E_{kb}(T_g)]; \quad i \neq j \quad (27)$$

where

$$\Lambda_{ji} = \frac{\rho_j F_{ji}}{D}, \quad \Lambda_{ii} = \frac{1-\rho_j F_{jj}}{D}$$

$$D = [1-\rho_1 F_{11}][1-\rho_2 F_{22}] - \rho_1\rho_2 F_{12} F_{21}$$

and, at first order

$$\phi_{kjm} = \rho_1 \Lambda_{j1}[Sa_{k11}\psi^{(1)}_{1m} + Sa_{k12}\psi^{(1)}_{k2m}] + \rho_2 \Lambda_{j2}[Sa_{k21}\psi^{(2)}_{1m} + Sa_{k22}\psi^{(2)}_{k2m}].$$

The above result would be particularly useful for the model of an industrial furnace consisting of a gas zone, a radiatively adiabatic zone and a radiative flux zone [10, 11].

The general enclosure

The most general enclosure, within the present constraints, consists of the gas zone and an arbitrary number of concave surfaces. The coefficient matrix for the band radiosities contains N^{2M+1} elements where N is the number of surfaces and M the order of the approximation. The maximum number of non-zero elements in any row is $N+1$ so that higher-order closure equations generally contain a large number of zero elements in the coefficient matrix. If the band radiosities are ordered so that the last superscript is varied most rapidly and the subscript the least rapidly the known constant vector contains N^{M+1} elements but the first N^M are identical as are all following groups of N^M, thus, there are N groups of N^M elements each. Consequently, when the coefficient matrix is inverted the N^{M+1} elements of each row can be arranged into N successive groups, each containing the sum of N^M elements.

The general solution for the band radiosities is, therefore, given by

$$R_{ki}^{(lmn\ldots st)} - E_{kb}(T_g) = \sum_{j=1}^{N} \psi_{kij}^{(lmn\ldots st)} \varepsilon_j [E_{kb}(T_j) - E_{kb}(T_g)]. \tag{28}$$

The next lower-order solution is

$$\begin{aligned} R_{kl}^{(mn\ldots st)} - E_{kb}(T_g) &= \varepsilon_l [E_{kb}(T_l) - E_{kb}(T_g)] \\ &+ \rho_l \sum_{i=1}^{N} \sum_{j=1}^{N} S_{kli}^{(mn\ldots st)} \psi_{kij}^{(lmn\ldots st)} \varepsilon_j [E_{kb}(T_j) - E_{kb}(T_g)] \end{aligned} \tag{29}$$

and the next is

$$\begin{aligned} R_{km}^{(n\ldots st)} - E_{kb}(T_g) &= \varepsilon_m [E_{kb}(T_m) - E_{kb}(T_g)] \\ &+ \rho_m \sum_{l=1}^{N} S_{kml}^{(n\ldots st)} \varepsilon_l [E_{kb}(T_l) - E_{kb}(T_g)] \\ &+ \rho_m \sum_{l=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_l S_{kml}^{(n\ldots st)} S_{kli}^{(mn\ldots st)} \psi_{kij}^{(lmn\ldots st)} \\ &\times \varepsilon_j [E_{kb}(T_j) - E_{kb}(T_g)]. \end{aligned} \tag{30}$$

The equations given above have considered the unrealistic case of concave surfaces in their formulation. It has previously been pointed out that plane or convex surfaces create combinatorial radiation paths that are physically meaningless. The most difficult combinatorial problem would attempt to eliminate these paths from all those which are possible. This is the direct elimination procedure previously mentioned. The indirect procedure, which accomplishes the same end, is to retain all radiosity functions which are combinatorially possible but to assign zero values to all band transmissivities which involve physically meaningless paths. This is certainly the easiest approach although it may prove to be computationally wasteful in some situations. With these guidelines in mind one can consider specific solutions from equations (28) to (30).

A first-order solution is obtained from equation (28) by deleting from the superscripts the group $(lmn\ldots s)$. A second-order solution is found from equation (29) by deleting the group $(mn\ldots s)$ and the third-order solution is given by equation (30) with the superscript group $(n\ldots s)$ deleted.

Solutions for the general enclosure are too complex to be solved without the aid of a computer. Solutions for three-zone enclosures can, in certain cases, be carried out with a simple calculator and in others with a programmable calculator. Example problems in this class will be considered later.

Since the band transmissivities are newly defined and essential to the present method of analysis, their evaluation will be considered in the following section.

EVALUATION OF BAND TRANSMISSIVITIES

Band transmissivities are essential numerical coefficients in the present method of analysis. Their evaluation is anything but a trivial effort and often is much too difficult to warrant an exact algorithm. This problem is characteristic of the zone method for an enclosure whether the confined medium is gray or otherwise. Fortunately, the approximate method based upon the concept of the mean beam length is very effective. This is particularly true for nongray media where it can be easily demonstrated that errors in the worst circumstances are less than 10% and generally much smaller for typical configurations of complex enclosures.

Approximate evaluation of band transmissivities

The band transmissivities were defined earlier in terms of spectral integrations. These can be very useful in certain circumstances but in general the necessary knowledge of spectral variations will neither be known nor necessary to a successful model of nongray absorption in terms of band absorption functions. By applying well-known definitions one can express a typical band transmissivity of first order in the form

$$\begin{aligned} Sa_{kti} S_{kij}^{(t)} &= \int_{\tilde{A}_i} \int_{\tilde{A}_t} \int_{\tilde{A}_i} \int_{\tilde{A}_j} [A(\tau_{kti} + \tau_{kij}) - A(\tau_{kij})] \\ &\times \frac{\cos\theta_i \cos\theta_t \cos\theta_i' \cos\theta_j}{\pi^2 r_{ti}^2 r_{ij}^2 \tilde{A}_t \tilde{A}_i} \, \mathrm{d}\tilde{A}_j \, \mathrm{d}\tilde{A}_i' \, \mathrm{d}\tilde{A}_t \, \mathrm{d}\tilde{A}_i \end{aligned} \tag{31}$$

which, as already pointed out, is somewhat complex for numerical quadrature. Higher-order transmissivities are even worse because each additional order requires the addition of two more surface integrations (Monte Carlo methods for integral evaluation could become very useful under these circumstances).

The mean-beam-length (MBL) approximation provides a neat way out of the obviously undesirable problem of evaluating equation (31). In this approximation one replaces the arguments τ_{kti} and τ_{kij}, which in general depend upon all the integration variables, by new arguments $u_{kti}\tau_{kti}$ and $u_{kij}\tau_{kij}$ which are independent of any integration variables. Consequently, one can then remove the band absorption functions from the integration and write

$$Sa_{kti}S^{(t)}_{kij} \approx F_{ti}F_{ij}[A(u_{kti}\tau_{kti}+u_{kij}\tau_{kij})-A(u_{kij}\tau_{kij})] \quad (32)$$

where the F_{ij} are the ordinary shape factors. In a very strict sense the above equation can be considered a definition of u_{kij} and u_{kti} through the relations

$$Sa_{kij} = F_{ij}A(u_{kij}\tau_{kij})$$
$$= \int_{\tilde{A}_i}\int_{\tilde{A}_j} A(\tau_{kij})\frac{\cos\theta_i\cos\theta_j}{\pi r_{ij}^2\tilde{A}_i}\,\mathrm{d}\tilde{A}_i\,\mathrm{d}\tilde{A}_j \quad (33)$$

and

$$F_{ti}F_{ij}A(u_{kti}\tau_{kti}+u_{kij}\tau_{kij})$$
$$= \int_{\tilde{A}_i}\int_{\tilde{A}_t}\int_{\tilde{A}_i}\int_{\tilde{A}_j} A(\tau_{kti}+\tau_{kij})$$
$$\times\frac{\cos\theta_i\cos\theta_t\cos\theta_i'\cos\theta_j}{\tilde{A}_t\tilde{A}_i\pi^2r_{ti}^2r_{ij}^2}\,\mathrm{d}\tilde{A}_j\,\mathrm{d}\tilde{A}_i'\,\mathrm{d}\tilde{A}_t\,\mathrm{d}\tilde{A}_i. \quad (34)$$

Equation (33) is suitable for the evaluation of u_{kij} but equation (34) is just as complex as the original equation from which it was derived. Consequently, one can define u_{kti} by the simpler requirement that

$$F_{ti}A(u_{kti}\tau_{kti}) = Sa_{kti}$$
$$= \int_{\tilde{A}_t}\int_{\tilde{A}_i} A(\tau_{kti})\frac{\cos\theta_t\cos\theta_i}{\tilde{A}_t\pi r_{ti}^2}\,\mathrm{d}\tilde{A}_i\,\mathrm{d}\tilde{A}_t \quad (35)$$

which is identical to the definition given for u_{kij} in equation (33). Equations (33) and (35) have been previously discussed by Dunkle [12, 13] who evaluated u_{kij} for simply arranged rectangular surfaces in the optically thin limit. As a practical matter one will ordinarily use the optically thin mean beam length for all values of optical depth because the evaluation for other circumstances is too complex for the few percent gain in accuracy. In this limit, equation (34) is simply expressed in terms of equations (33) and (35). Furthermore, in the optically thin limit, u_{kij} is independent of band parameters so that $u_{kij} = u_{ij}$ and the simplest MBL approximation can be written as

$$S^{(t)}_{kij} = \frac{F_{ij}[A(u_{ti}\tau_{kti}+u_{ij}\tau_{kij})-A(u_{ij}\tau_{kij})]}{A(u_{ti}\tau_{kti})} \quad (36)$$

where the beam length factors u are found in the way outlined by Dunkle [12].

Similar arguments lead to the second-order approximation

$$S^{(it)}_{kjm} =$$
$$\frac{F_{jm}[A(u_{ti}\tau_{kti}+u_{ij}\tau_{kij}+u_{jm}\tau_{kjm})-A(u_{ij}\tau_{kij}+u_{jm}\tau_{kjm})]}{A(u_{ti}\tau_{kti}+u_{ij}\tau_{kij})-A(u_{ij}\tau_{kij})}. \quad (37)$$

All higher-order equations are of this form with extended arguments in the band absorption functions. It should be noticed that band transmissivities of second and higher order are fundamentally different in form from that of first order. For gray bands the difference is apparent and not real but for molecular gas absorption there is a clear and important effect [9] which will be discussed later.

For the simple geometries previously considered [9] it is possible to compare mean beam length and exact results. The differences are greatest for the planar medium, as would be intuitively expected, and so numerical comparisons are restricted to this geometry.

For the planar medium the approximate band transmissivities of any order n reduce to

$$S^{(n)}_k = \frac{A[(n+1)u\tau_k]-A[nu\tau_k]}{A[nu\tau_k]-A[(n-1)u\tau_k]} \quad (38)$$

where $u = 2$ is appropriate to the optically thin limit and τ_k is based upon the plate separation, h.

Equation (38) is very similar to the specular surface result obtained in ref. [14]. In Table 2, first-order MBL solutions for the single-band radiative flux from an isothermal gas is compared to the first-order results obtained earlier [9] in terms of the percent difference. The MBL results are always greater and this compensates to varying degree for the inherent underprediction of first-order solutions using exact exchange factors and band transmissivities. For wall reflectivities of 0.2 the MBL results overpredict the exact (infinite order) solution but at the important larger optical depths ($\tau > 10$) the overprediction is less than 5%. At a wall reflectivity of 0.8 the MBL results actually improve upon those obtained earlier [9] but the results are still lower than the exact at large optical depth by up to 10%.

Results for cylindrical and spherical geometries at first order show a much smaller difference between MBL and exact exchange factor methods, differences being a few percent for a reflectivity of 0.2 and a fraction of a percent at a reflectivity of 0.8.

Real enclosures will nearly always be more similar to cylindrical or spherical geometries and consequently the MBL approximation is not only justified but in most situations is the only practical approach to pursue. Of course, fundamental research will require that exact band transmissivities be eventually determined for selected geometrical configurations.

Table 2. The percentage change in first-order flux predictions for the planar two-zone enclosure due to the MBL approximation

τ	ρ 0.2	 0.5	 0.8
0.1	4.890	4.432	3.063
0.2	6.911	5.827	3.141
0.5	9.233	6.859	2.518
1.0	9.669	6.723	2.346
2.0	8.572	6.050	2.791
5.0	6.316	4.843	3.108
10.0	5.101	4.081	2.921
20.0	4.273	3.507	2.656
50.0	3.517	2.949	2.329
100.0	3.102	2.629	2.119
1000.0	2.228	1.928	1.613

Reciprocity relations

Reciprocity relations play an important role in the overall simplification of the evaluation of radiative exchange factors whether they are simple shape factors or the more complex functions defined in this paper.

The reciprocity equation for the band absorption factor Sa_{kij} is easily deduced from equation (35) to be

$$\tilde{A}_i Sa_{kij} = \tilde{A}_j Sa_{kji} \tag{39a}$$

or

$$F_{ji} Sa_{kij} = F_{ij} Sa_{kji} \tag{39b}$$

the first of which is clearly analogous to that for simple shape factors.

In the case of the first-order band transmissivity defined by equation (31) only the exchange of t and j makes any sense for reciprocity. For this case it amounts to reversing the path of the radiation and one obtains from equation (31)

$$F_{jt}[Sa_{kti} S^{(t)}_{kij} + F_{ti} Sa_{kij}] = F_{tj}[Sa_{kji} S^{(j)}_{kit} + F_{ji} Sa_{kit}] \tag{40}$$

or

$$Sa_{kit} S^{(t)}_{kij} + F_{it} Sa_{kij} = Sa_{kij} S^{(j)}_{kit} + F_{ij} Sa_{kit} \tag{41}$$

from which it follows that

$$Sa_{kit}[F_{ij} - S^{(t)}_{kij}] = Sa_{kij}[F_{it} - S^{(j)}_{kit}]. \tag{42}$$

This is basically an emission–absorption equivalence relation which says that the effective emission from surface j to surface i, which is then partially absorbed between surfaces i and t, is equal to the effective emission from surface t to i which is then partially absorbed between i and j, the emissive powers of surface j and t being equal to one another. In retrospect, this should be thermodynamically obvious but the condition in this particular form for nongray-band absorption/emission has not likely been stated before. Indeed, it seems remote that equation (42) could have been deduced directly from equilibrium thermodynamics of nongray radiation. Equation (42) is also a reciprocity relation which the above discussion might have obscured. It suffices to say that the reversed path band transmissivity function is obtainable from the forward path band transmissivity in accord with equation (42) or (40).

Equation (42) is a general nongray result which also applies for the special case of gray-band absorption. Gray-band results are not path dependent and, therefore, the superscripts are redundant. Consequently, for gray bands, one can rewrite equation (42) in the form

$$\frac{Sa_{kit}}{F_{it} - S_{kit}} = \frac{Sa_{kij}}{F_{ij} - S_{kij}}. \tag{43}$$

gray band

Since t and j can be varied independently, each side must be equal to a common constant. In fact, for a gray band, it is well known that [10]

$$Sa_{kit} = F_{it} - S_{kit} \tag{44}$$

gray band

but it is important to note that in general

$$Sa_{kit} \neq F_{it} - S^{(j)}_{kit}. \tag{45}$$

nongray band

Nevertheless, equation (42) can be expressed in the remarkable form

$$\frac{Sa_{kit}}{Sa_{kij}} = \frac{F_{it} - S^{(j)}_{kit}}{F_{ij} - S^{(t)}_{kij}}. \tag{46}$$

One can also find a connection between higher-order band transmissivities and their inverse. At second order one finds

$$\begin{aligned} &F_{mt}[Sa_{kti} S^{(t)}_{kij} S^{(it)}_{kjm} + F_{ti} Sa_{kij} S^{(i)}_{kjm} + F_{ti} F_{ij} Sa_{kjm}] \\ &= F_{tm}[Sa_{kmj} S^{(m)}_{kji} S^{(jm)}_{kit} + F_{mj} Sa_{kji} S^{(j)}_{kit} + F_{mj} F_{ji} Sa_{kit}]. \end{aligned} \tag{47}$$

From this result and that for first order the form for higher-order functions is clear. The recurrence relation between a band transmissivity and its inverse may be used to reduce computational evaluations by nearly a factor of two.

Special simplifications

In a remarkable number of cases the asymptotic forms of band absorption equations satisfy a power law or a logarithmic variation with optical depth. Narrow-band models such as those of Elsasser or Mayer-Goody are often used when relatively high spectral resolution is desirable. This is rarely necessary in heat transfer problems of engineering concern but the narrow-band models play an important role in the more global expressions of total band absorption as discussed by Edwards [7].

In the present context a considerable simplification results in the evaluation of exchange factors and band transmissivities for the particular mathematical forms mentioned above.

The important mathematical/physical relations for band absorption are linear, square-root and logarithmic [7]. Numerous publications, which are not mentioned here, have considered factors such as Sa_k leading to the realization that the important effect of power-law and logarithmic forms is that radiative properties and geometric properties are separable for this particular exchange factor. That is to say that integrations required in the evaluation of exchange factors are purely geometrical whenever power-law and logarithmic band absorption functions are valid.

This fact carries over to the band transmissivities. Whenever band parameters are appropriate to any of these limits, band transmissivities of higher order than the first are purely geometric functions. This can lead to a significant reduction in solution time requirements if properly exploited. To fix these ideas, the reader should consider equations (36) and (37) for the special cases: (1) $A(x) = x$; (2) $A(x) = \sqrt{(4\beta x)} - \beta$; and (3) $A(x) = \ln x + c$ while making use of the fact that $\tau_{kij} = \rho_k \alpha_k L_{ij}/\omega_k$.

APPLICATION OF PRINCIPLES

The foregoing nongray method is derived from fundamental principles and the approximate procedure called the 'closure method' has a clear physical interpretation which is independent of any particular band model.

To illustrate the method in a reasonably realistic context the three-zone enclosure consisting of a gas zone, an adiabatic zone, and a heat transfer zone will be considered. The easiest geometry is a cube. Simplicity in other respects suggest one surface at constant temperature and the rest as adiabatic. In a purely arbitrary fashion, the constant surface temperature is 750 K. The gas phase consists of $CO_2/H_2O/N_2$ with mole fractions of 0.15, 0.15, and 0.7, respectively. The gas temperature is 1500 K and the total pressure is 1 atm. The objective is to find the heat flux at the constant temperature surface. Practical considerations dictate the use of the MBL approximation.

Optically thin beam lengths taken from Dunkle's paper [12] are used. Five bands are considered; for CO_2 the 15, 4.3, and 2.7 μm and for H_2O the 6.3 and 2.7 μm. The optical depths for the CO_2 bands range from about 3 to 180 and all obey the overlapped line logarithmic equation. The 6.3 μm H_2O band is also logarithmic at an optical depth of about 5.5 but is nonoverlapped in structure (this is called the log-root equation by Edwards [7]). The 2.7 μm H_2O band is square-root and has an optical depth of about 3.

First- and second-order solutions are considered. Band radiosities of interest are $R_{k1}^{(2)}$, $R_{k2}^{(1)}$, $R_{k2}^{(2)}$ where surface 1 is the planar heat transfer surface.

For this geometry all beam lengths are found to be equal to the single value of $2L/3$ where L is the length of one side of the cube ($L = 2m$). All of the band radiation properties can be determined from Edwards' work [7]. Knowledge of beam lengths and band radiation properties is all that is needed to find exchange factors and band transmissivities for any order of solution.

Analysis of this problem is greatly simplified by the fact that all bands are either logarithmic or square-root. This simplification is quite apart from the MBL approach and would apply even if exact exchange factors and band transmissivities were used. However, use of the MBL approximation greatly simplified the solution because the band transmissivities only depend upon the subscripts through the ordinary shape factor and superscripts are irrelevant for this geometry.

The present problem requires an iteration to find the unknown temperature of the adiabatic zone. Satisfactory convergence was found in about four tries using an unsophisticated approach with a programmable calculator of modest capabilities. Both first- and second-order solutions were carried out without the aid of a computer. Considerable algebraic effort was required because first-order solutions required the inversion of a 4 × 4 matrix and second-order solutions involved an 8 × 8 matrix. However, the actual enclosure considered did not require such large matrices because of the planar surface assumption.

Table 3 presents results for $\rho_1 = 0.2$ and $\rho_2 = 0.5$ for both first- and second-order solutions. Results are also given for the black wall enclosure to fix a frame of reference.

The most important result of this example is the small difference between the first- and second-order solutions for the heat flux. This is consistent with earlier results for a very simple two-zone problem [9] and suggests that the error in the second-order solution is very likely less than 1%. The error referred to deals specifically with nongray-absorption effects and is quite aside from the discretization error of the zone method or of the MBL approximation.

The ability to gauge the level of error at any order of approximation by performing the next higher-order solution is a particularly important feature of the present method. The band model is completely arbitrary and, in fact, different models can be used for different bands. One can use narrow-band (quasi-spectral) models or total-band models or, if desired, one could mix these in any convenient way. Since band absorption properties are used directly, there is no need to perform calculations of total radiation properties such as emissivity and absorptivity and, subsequently, curve-fitting such results to conform to a gray-band model. The present method uses direct, band-exchange factors exclusively and, as a consequence of the constant-property assumption, these are independent of the actual temperatures within the enclosure. Exchange factors which account for the effects of all bands are necessarily temperature dependent, even for constant radiation properties, and should be avoided for that reason alone.

It has been realized for some time that total-radiation properties alone might be inadequate for radiative transfer analysis and that one should preserve spectral information at least at the level of total-band absorption. The present method is a completely satisfactory approach to the theoretical problem of incorporating band radiation models into arbitrary, diffuse-walled enclosure analysis. At first exposure, it

Table 3. Heat transfer results for a three-zone enclosure containing an isothermal gas

Wall reflectivity	Solution method	Heat flux (W m^{-2})	Adiabatic wall temperature (K)
$\rho_1 = 0.2$	First order	-1.269×10^5	$T_2 = 1298$
$\rho_2 = 0.5$	Second order	-1.291×10^5	1302.5
$\rho_1 = \rho_2 = 0$	—	-1.712×10^5	1309.7

might appear somewhat complex but in fact it is simpler than other, less exact, alternative methods and has unique advantages. Its greatest strength lies in the fact that it is derived from fundamental principles with the least number of physical assumptions presently accepted (band models) and that the assumption leading to the closure method has a clear physical interpretation.

CONCLUDING REMARKS

There is, and has been for some time, a clear need for a rigorous treatment of nongray radiative transfer within diffuse-walled but otherwise arbitrary enclosures. The generally accepted and well-known method of Hottel has always been suspect because of its gray-band foundations. Although it is natural to extend a well-understood method to more complex situations, it has been found in a number of independent studies that the nongray behavior of molecular gases is difficult, if not impossible, to describe by gray-band models.

The motivation for using gray-band models has largely disappeared over the past 20 years. Band-radiation properties are now reasonably well understood and it is common to use this knowledge to compute total-radiation properties required by the gray-band models. It seems reasonable to bypass this tedious procedure altogether and use band-radiation properties directly in the analytical method.

The method developed in this paper has a number of attractive features but does, in some respects, appear somewhat complicated. At the same time it displays certain remarkable simplifications when applied to molecular gas band radiation which might very well prove it to be simpler to apply in practical situations than the gray-band alternative.

At the very least, the method can be used to test other procedures because of its fundamental foundations. Its generalization to nonisothermal gases is straightforward and will be reported in the future.

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APPENDIX

Equation (1) applies only to an enclosure which contains an isothermal gas and does not follow directly from a simple inspection of results from ref. [9]. The derivation follows.

From ref. [9] the gas cooling rate for a single gas zone (g) is

$$Q^{*r} = V \sum_{k=1}^{K} 4\rho_k \alpha_k \sum_{i=1}^{N} Gs_{kgi}[E_{kb}(T_g) - \tilde{R}_{ki}^{(g)}]$$

$$= \sum_{k=1}^{K} \sum_{i=1}^{N} \omega_k \tilde{A}_i Sg_{kig}[E_{kb}(T_g) - \tilde{R}_{ki}^{(g)}]$$

which follows from equation (21) of ref. [9]. The definition of $\tilde{R}_{ki}^{(g)}$ from ref. [9] can be written as

$$\omega_k Sg_{kig} \tilde{R}_{ki}^{(g)} = \int_{\Delta\nu_k} Sg_{\nu ig} R_{\nu i} \, d\nu$$

in which $Sg_{\nu ig} = (s_i g_g)_\nu / A_i$ and $4\rho_k \alpha_k V Gs_{kgi} = \omega_k \tilde{A}_i Sg_{kig}$ have

$$Sg_{\nu ig} = \sum_{t=1}^{N} Sa_{\nu it}$$

one can write

$$\omega_k Sg_{kig} \tilde{R}_{ki}^{(g)} = \sum_{t=1}^{N} \int_{\Delta\nu_k} Sa_{\nu it} R_{\nu i} \, d\nu = \sum_{t=1}^{N} \omega_k Sa_{kit} R_{ki}^{(t)}$$

which leads directly to equation (1) of the text by using in addition

$$Sg_{kig} = \sum_{t=1}^{N} Sa_{kit}.$$

This result applies only for an isothermal gas zone.

It was mentioned in the text that the coefficients for the three-zone enclosure satisfied a symmetry condition which made it possible to find eight values by listing only four of them. Since the problem in question only involves two surfaces, the rule is simply an exchange of indices, that is all 1's are replaced by 2's and vice versa. The four coefficients are arbitrarily given as

$$\psi^{(1)}_{k11} = \{1-\rho_2 S^{(2)}_{k22}-\rho_1 S^{(2)}_{12}[\rho_2 S^{(1)}_{k21}[1-\rho_2 S^{(2)}_{k22}]$$
$$+\rho_2\rho_2 S^{(1)}_{k22}S^{(2)}_{k21}]$$
$$+\rho_1 S^{(1)}_{k12}[\rho_2 S^{(1)}_{k21}[1-\rho_2 S^{(2)}_{k22}]$$
$$+\rho_2\rho_2 S^{(1)}_{k22}S^{(2)}_{k21}]\}D^{-1}$$

$$\psi^{(1)}_{k12} = \{\rho_1 S^{(1)}_{k12}[1-\rho_2 S^{(2)}_{k22}]+\rho_1\rho_2 S^{(1)}_{k12}S^{(1)}_{k22}\}D^{-1}$$

$$\psi^{(2)}_{k11} = \{\rho_1 S^{(2)}_{k11}[1-\rho_2 S^{(2)}_{k22}]+[1-\rho_1 S^{(1)}_{k11}][1-\rho_2 S^{(2)}_{k22}]\}D^{-1}$$

$$\psi^{(2)}_{k12} = \{[1-\rho_2 S^{(2)}_{k22}][\rho_1 S^{(2)}_{k12}[1-\rho_1 S^{(1)}_{k11}]+\rho_1\rho_1 S^{(1)}_{k12}S^{(2)}_{k11}]$$
$$+\rho_2 S^{(1)}_{k22}[\rho_1 S^{(2)}_{k12}[1-\rho_1 S^{(1)}_{k11}]+\rho_1\rho_1 S^{(1)}_{k12}S^{(2)}_{k11}]\}D^{-1}$$

where

$$D = [1-\rho_1 S^{(1)}_{k11}][1-\rho_2 S^{(2)}_{k22}]$$
$$-\rho_2 S^{(1)}_{k21}[1-\rho_2 S^{(2)}_{k22}]\{[1-\rho_1 S^{(1)}_{11}]\rho_1 S^{(2)}_{k12}+\rho_1\rho_1 S^{(2)}_{k11}S^{(1)}_{k12}\}$$
$$-\rho_2\rho_2 S^{(2)}_{k21}S^{(1)}_{k22}\{[1-\rho_1 S^{(1)}_{k11}]\rho_1 S^{(2)}_{k12}+\rho_1\rho_1 S^{(2)}_{k11}S^{(1)}_{k12}\}$$

and one should not exchange indices in this relation.

RADIATION PAR BANDES DES GAZ ISOTHERMES DANS DES ENCEINTES A PAROI DIFFUSE

Résumé—On présente une méthode d'obtention de solutions très précises des équations du transfert radiatif, intégrées sur le spectre, pour des enceintes quelconques qui contiennent un gaz isotherme. La procédure est assez générale pour être appliquée à un spectre d'absorption–émission non gris quelconque mais il est particulièrement approprié pour le rayonnement par bandes de gaz moléculaires. Cela constitue une alternative raisonnée aux procédures de bandes grises on de gaz gris actuellement existantes.

BANDSTRAHLUNG ISOTHERMER GASE INNERHALB VON HOHLRÄUMEN MIT DIFFUSEN WÄNDEN

Zusammenfassung—In dieser Arbeit wird eine Methode zur Gewinnung sehr genauer Näherungslösungen der spektral integrierten, zonalen Strahlungstransportgleichungen für beliebige Hohlräume, die ein nicht graues, isothermes Gas enthalten, entwickelt. Das Verfahren ist ausreichend allgemein, um auf jedes nichtgraue Absorptions–Emissionsspektrum angewendet werden zu können, ist aber besonders für die Bandstrahlung molekularer Gase geeignet. Es stellt eine fundamental begründete Alternative zu den gegenwärtig verfügbaren Prozeduren zur Berechnung der grauen Band- bzw. Gasstrahlung dar.

ИЗЛУЧЕНИЕ В ПОЛОСАХ ИЗОТЕРМИЧЕСКИХ ГАЗОВ В ПОЛОСТЯХ С ДИФФУЗНЫМИ СТЕНКАМИ

Аннотация—Предложен метод получения довольно точных приближенных решений уравнений интегрального по спектру зонного лучистого переноса в полостях произвольной формы, содержащих несерый изотермический газ. Методика является достаточно общей и может использоваться для анализа абсорбционно-эмиссионного спектра любой несерой среды, но больше всего подходит для решения задач излучения в полосах молекулярных газов. Она представляет собой фундаментально обоснованный вариант существующих методов серых полос или серого газа.